Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

INVESTIGATION OF A CLASS OF FUNDAMENTAL INEQUALITIES IN THE THEORY OF ANALYTIC FUNCTIONS.

BY J. L. W. V. JENSEN.

(Authorized translation from the Danish* by T. H. GRONWALL.)

TABLE OF CONTENTS.

PAGE.

§ 1. Introduction 1

§ 2. Formulas and rules for reckoning with ordinary complex numbers. Notations, definitions, and lemmas 6

§ 3. General theorems concerning functions bounded with respect to their absolute value .. 12

§ 4. General theorems concerning functions bounded with respect to their real part 22

§ 1. Introduction.

In recent years, a class of function theoretic problems have acquired a steadily increasing importance, viz. problems of the following nature. We consider an otherwise arbitrary analytic function $y = y(x)$ of the complex variable x, and this function y is assumed to be holomorphic inside the circle $| x | < R$; in other words, we consider an arbitrary power series in x with a radius of convergence $\geqq R$. Moreover, we assume regarding this function that $| y | < M$, or $\mathfrak{R}(y) < A$, or similar conditions,† all this for $| x | < R$, and finally that we know the value of the function at a given point inside the circle, usually the center, i.e., $y(0)$ is known. What further general statements may then be made regarding the function, what may be said of its absolute value, of its real or its imaginary part, of its zeros or of regions where there are no zeros, of its derivative, and so forth?

Before presenting my own investigations, I shall review briefly (and to the extent of my knowledge of the literature on the subject) the more or less special results in this direction obtained by those mathematicians who have made substantial contributions to the solution of problems of this nature. To facilitate comparison, I shall *use greatly changed notations throughout*, which are in accord with those I use later on.

---

* Undersögelser over en Klasse fundamentale Uligheder i de analytiske Funktioners Theori. D. Kgl. Danske Vidensk. Selsk. Skrifter, Naturv. og Mathematisk Afd., ser. 8, vol. 2, p. 199–228 (1916). Essential parts of this paper were communicated earlier in two lectures, one on the occasion of presenting these investigations to the Royal Danish Academy of Sciences at the meeting Nov. 19, 1915, and another, previous to this, before the Copenhagen Mathematical Society Sept. 16, 1915, on reckoning with ordinary complex numbers with applications to function theory.

† For the notations, see § 2.

1. (1869.) In the first place, mention should be made of a proposition due to Schwarz[1] which may be stated as follows: When y, besides being holomorphic and less than M in absolute value inside the circle $|x| < R$, has the property that $y(0) = 0$, then we have for $|x| < R$

$$|y| \leqq \frac{M|x|}{R}.$$

Carathéodory[2] was probably first in emphasizing the great importance of this proposition, and proved it in a very simple and elementary manner. The name "Schwarz' lemma" introduced by this author will be adhered to in the following.

2. (1892.) Toward the discovery of properties of y inside the circle frequently referred to, when a positive upper bound of the real part of the function is given, Hadamard[3] made a contribution which led Borel[4] to the following solution of a problem of this kind:

$$|y| < |\Re y(0)| + |\Im y(0)| + (4A + 2|\Re y(0)|)\frac{|x|}{R - |x|},$$

where A denotes an upper (positive) bound of the real part $\Re(y)$ of y, and $\Im(y)$ is the "imaginary part" of y (or more correctly expressed, the coefficient of i in the purely imaginary part). Schottky[5] improved upon this solution by means of the following formula obtained from the Cauchy integral

$$|y| \leqq |\Im y(0)| + \frac{1}{2\pi}\int_0^{2\pi} |\Re y(re^{i\theta})|\, d\theta \cdot \frac{r + |x|}{r - |x|},$$

where $|x| < r < R$. In accordance with the line of reasoning in the paper quoted in the footnote, this formula gives, with the notations used above

$$|y| \leqq |\Im y(0)| + (2A - \Re y(0))\frac{R + |x|}{R - |x|}.$$

---

[1] H. A. Schwarz, "Zur Theorie der Abbildung." Programm der eidgenössischen polytechnischen Schule in Zürich für das Schuljahr 1869–70; Gesammelte Mathematische Abhandlungen (Berlin, 1890), vol. 2, p. 108–132 (see p. 110–111).

[2] C. Carathéodory, "Sur quelques généralisations du théorème de M. Picard," Comptes rendus de l'Académie des Sciences, vol. 141, Paris, 1905, p. 1213–1215.

E. Landau, "Ueber den Picardschen Satz," Vierteljahrsschrift der naturforschenden Gesellschaft in Zürich, vol. 51, 1906, p. 252–318 (see p. 271).

C. Carathéodory, "Untersuchungen über die konformen Abbildungen von festen und veränderlichen Gebieten," Mathematische Annalen, vol. 72, 1912, p. 107–144 (see p. 110).

[3] J. Hadamard, "Sur les fonctions entières de la forme $e^{G(x)}$," Comptes rendus, vol. 114, 1892, p. 1053–1055.

[4] E. Borel, "Démonstration élémentaire d'un théorème de M. Picard sur les fonctions entières," Comptes rendus, vol. 122, 1896, p. 1045–1048.

[5] F. Schottky, "Ueber den Picard'schen Satz und die Borel'schen Ungleichungen," Sitzungsberichte der Kgl. Preussischen Akademie der Wissenschaften, 1904, p. 1244–1262 (see p. 1246–1247).

Carathéodory[6] improved this to

$$| y | \leqq | \Im y(0) | + 2A \frac{| x |}{R - | x |} + | \Re y(0) | \frac{R + | x |}{R - | x |},$$

where A is no longer assumed to be positive. This formula is commonly called "Carathéodory's theorem."

3. (1899.) Regarding the zeros of y the author[7] gave a result which may be stated as follows: When y is holomorphic for $| x | < R$, and $| y | < M$, and when $x_1, x_2, \cdots, x_n$ denote those zeros of y which fall inside or on the circumference $| x | = r < R$, we have

$$M > | y(0) | \frac{R^n}{| x_1 x_2 \cdots x_n |}.$$

Here it is evident that y has no zeros for

$$| x | < \frac{| y(0) |}{M} R.$$

An improvement on this proposition is due to Carathéodory and Fejér[8] who, besides giving a new proof, showed that the equality sign will occur in the inequality above when y belongs to a certain class of rational functions of x. (Incidentally I may state that this note together with my own methods for easy reckoning with complex numbers have formed an essential reason for the publication of the present paper.)

A somewhat more general result has been obtained recently by Landau[9] who shows that when η is a constant less than M in absolute value, and $x_1, x_2, \cdots, x_n$ are zeros of the function $y - \eta$ inside $| x | = R$, and η and $\bar{\eta}$ denote conjugate numbers, then

$$\left| \frac{M^2 - \bar{\eta} y(0)}{M(y(0) - \eta)} \right| \geqq \frac{R^n}{| x_1 x_2 \cdots x_n |},$$

whence it may be shown that

$$y \neq \eta \qquad \text{for} \qquad | x | < \left| \frac{M(y(0) - \eta)}{M^2 - \bar{\eta} y(0)} \right| R.$$

Moreover, Landau gives a direct proof of the last proposition.

---

[6] E. Landau, l. c.,[2] p. 277. See also E. Landau, "Beiträge zur analytischen Zahlentheorie," Rendiconti del Circolo Matematico di Palermo, vol. 26, 1908, p. 169–302 (see p. 191–193).

[7] J. L. W. V. Jensen, "Sur un nouvel et important théorème de la théorie des fonctions," Acta Mathematica, vol. 22, 1899, p. 359–364 (see p. 362–363).

The author also presented this formula in a lecture before the Copenhagen Mathematical Society three years earlier. The proof given here was entirely elementary.

[8] C. Carathéodory et L. Fejér, "Remarques sur le théorème de M. Jensen," Comptes rendus, vol. 145, 1907, p. 163–165.

[9] E. Landau, "Ueber eine Aufgabe aus der Funktionentheorie," The Tôhoku Mathematical Journal, vol. 5, 1914, p. 97–116 (see p. 107 and p. 105).

4. (1906.) In regard to $y'(x)$, the derivative of y, the first result is due to Landau (l. c.,[2] p. 305–306) who shows that when $| y | < M$ for $| x | < R$, then

$$| y'(0) | \leqq \frac{M^2 - | y(0) |^2}{RM}.$$

A corresponding result obtained by F. W. Wiener[10] may be written as follows:

$$| y^{(n)}(0) | \leqq n! \frac{M^2 - | y(0) |^2}{R^n M}.$$

5. (1908.) The most comprehensive set of results concerning the problems treated here was given however by Lindelöf;[11] by means of the theory of Green's functions and conformal mapping he sets up a principle which, as he shows at length, gives the solution of a whole set of problems containing those mentioned in 1, 2, and 4 as special cases. To these are added new, elegant and important results, such as an extension of Schwarz' lemma

$$| y | \leqq M \frac{R | y(0) | + M | x |}{RM + | y(0) x |} \qquad \text{when} \qquad | y | < M \text{ for } | x | < R,$$

an elegant extension of Landau's formula above

$$| y'(x) | \leqq \frac{R}{M} \frac{M^2 - | y |^2}{R^2 - | x |^2},$$

and others, not to mention the determination of a region in the circle $| x | < R$ and in the vicinity of an arbitrary point x_0, in which region $y \neq 0$. He does not consider, however, the other problems falling under 3, and he restricts himself, as do most of the other authors, to assuming $y(0)$ as the known value of y instead of making the easy extension to the case when $y_0 = y(x_0)$ is assumed as the known value, x_0 being fixed arbitrarily. In the following, I shall have occasion to mention the results of Lindelöf individually, since they will occur as special cases of other and more general theorems.

6. (1914.) Carathéodory[12] was probably first in giving a formula for

---

[10] H. Bohr, "A theorem concerning power series," Proceedings of the London Mathematical Society, ser. 2, vol. 13, 1913 (see the lemma used in this paper).

[11] E. Lindelöf, "Mémoire sur certaines inégalités dans la théorie des fonctions monogènes et sur quelques propriétés nouvelles de ces fonctions dans le voisinage d'un point singulier essentiel," Acta Societatis Scientiarum Fennicae, vol. 25, no. 7, p. 1–35.

[12] C. Carathéodory, "Elementarer Beweis für den Fundamentalsatz der konformen Abbildungen," Mathematische Abhandlungen Hermann Amandus Schwarz zu seinem fünfzigjährigen Doktorjubiläum am 6. August 1914 gewidmet, Berlin, 1914, p. 19–41 (see p. 23).

the difference quotient, viz.,

$$\frac{|y - y_0|}{|x - x_0|} \leqq \frac{R\,|M^2 - \bar{y}_0 y|}{M\,|R^2 - \bar{x}_0 x|} < \frac{2MR}{R^2 - |x_0 x|}.$$

Regarding these formulas, we may note that the first inequality gives Lindelöf's formula above when $x \rightarrow x_0$; the second is only a crude approximation fully sufficient, however, for the purposes of the paper quoted.

A close examination of these solutions of special problems reveals a lack of unity in treatment which it seems desirable to remove. In the present paper, I have therefore worked these theorems into a connected whole and generalized them considerably, so that my results contain all the previous ones as very special cases, as well as a number of new propositions, no special cases of which were known previously; I have also attempted to do this in the simplest and most elementary manner. I hope that it will appear from the following that it will hardly be possible to reduce the solutions of these problems to a higher degree of simplicity, or base them on a smaller number of function theoretic propositions, all of the latter belonging moreover to that part of the elementary theory of functions which is most elegantly treated by means of power series.

It is almost superfluous to state that the celebrated theorems of Picard, or rather the elementary proofs and generalizations due to Borel, Landau and Schottky, are intimately connected with some of our problems, particularly those mentioned in 2. This is apparent moreover from the titles quoted under [2, 4, 5] It is however beyond the scope of the present paper to consider applications of the theorems presented.

Before proceeding to those rules for reckoning with ordinary complex numbers which form the greater part of the lemmas used in the following (and which could surely find their place in a textbook on elementary algebra), I wish to make a remark on the problems treated and their solutions. In these, the inequalities which together with certain simple assumptions form the basis of the solution, are used to derive other inequalities, frequently altogether different in form. In certain cases, and often in the most important problems, it is possible by an algebraic transformation of the final inequalities to return a fortiori to precisely the initial conditions. To make this clear in the very simplest case, let us consider Schwarz' lemma. From the assumptions, we derive the solution $|y| \leqq M|x|/R$; from this inequality it follows at once that $y(0) = 0$ and that, a fortiori, $|y| < M$, which are exactly the assumptions from which we started; moreover, the solution contains an equality sign which cannot be dispensed with. When these circumstances occur in the solution of a problem, as will be the case continually in the following, we say that the solution is *complete.* A final remark on inequalities is this: When an inequality $A_1 \leqq B_1$ leads to another $A_2 \leqq B_2$ by a finite number of algebraic operations, and *vice versa,* and when moreover an inequality sign corresponds to an inequality sign, a sign of equality to a sign of equality, these two inequalities are said to be *equivalent.* On the contrary, when the first inequality does not follow from the second by operations of the nature indicated above, the second inequality is said to follow *a fortiori* from the first.

§ 2. Formulas and rules for reckoning with ordinary complex numbers. Notations, definitions, and lemmas.

In the following, all numbers are ordinary complex numbers unless the contrary is expressly stated. As usual, u and $\bar{u}$ denote conjugate numbers, and $|u|$ the absolute value of u; we then have $u\bar{u} = |u|^2$. By $\Re(u) = \frac{1}{2}(u + \bar{u})$ we denote the real part of u, and by

$$\Im(u) = \frac{1}{2i}(u - \bar{u}) = \Re\left(\frac{u}{i}\right)$$

the coefficient of i in the purely imaginary part of u (read, for brevity, "imaginary part of u"), or

$$u = \Re(u) + i\Im(u).$$

Occasionally we also use the notation

$$\operatorname{sg}(u) = \frac{u}{|u|} \quad \text{(read "signum } u\text{")}$$

when $u \neq 0$, while $\operatorname{sg}(0) = 0$.

With these notations, there follows from $|u + v|^2 = (u + v)(\bar{u} + \bar{v})$ the identity

$$|u + v|^2 = |u|^2 + 2\Re(u\bar{v}) + |v|^2$$[*]

which is of importance in the following. This identity leads to another

$$|\beta u + \alpha v|^2 - |\bar{\alpha} u + \bar{\beta} v|^2 = (|\beta|^2 - |\alpha|^2)(|u|^2 - |v|^2)$$[†]

---

[*] For easy reckoning with complex numbers the following identities are also useful

$$|u + v|^2 - |u - v|^2 = 4\Re(u\bar{v}) \qquad \text{or} \qquad |u|^2 - |v|^2 = \Re((u + v)(\bar{u} - \bar{v})),$$

$$|u + v|^2 - |u - \bar{v}|^2 = 4\Re(u)\Re(v) = |u + \bar{v}|^2 - |u - v|^2,$$

$$|u + v|^2 - |u + \bar{v}|^2 = 4\Im(u)\Im(v);$$

however, these identities will not be needed in the present paper.

[†] Other identities are

$$|\beta u + \alpha v|^2 + |\bar{\alpha} u - \bar{\beta} v|^2 = (|\alpha|^2 + |\beta|^2)(|u|^2 + |v|^2),$$

$$|\beta u + \alpha v|^2 - |\bar{\beta} u - \bar{\alpha} v|^2 = 4\Re(\alpha\bar{\beta})\Re(u\bar{v}),$$

$$|\beta u + \alpha v|^2 - |\bar{\beta} u + \bar{\alpha} v|^2 = 4\Im(\alpha\bar{\beta})\Im(u\bar{v}).$$

simply by observing that the terms in $\mathfrak{R}$ from the two squares to the left are identical. From the last identity, we shall now derive all the algebraic lemmas we need. It should be noted that since the right side of the identity contains only the absolute values of the four variables, we may change the arguments of α, β, u, v in any manner in the expression to the left without changing its value.

LEMMA 1. *When the complex number* α *is such that* $|\alpha| < 1$, *the following inequalities, in which inequality and equality signs read from top to bottom correspond to each other,*

$$|u| \lesseqqgtr |v| \qquad \text{and} \qquad |u + \alpha v| \lesseqqgtr |\bar{\alpha}u + v|$$

are equivalent.

When $|\alpha| > 1$, *the same statement is true provided that one of the inequalities is reversed, i.e., the signs from top to bottom are read in the reversed order.*

When finally $|\alpha| = 1$, *we always have* $|u + \alpha v| = |\bar{\alpha}u + v|$.

THE PROOF is read off at once from our identity by making $\beta = 1$ and comparing the signs of both sides.

NOTE. This lemma, which we shall also call *the* α *method,* is very useful for transforming inequalities between the absolute values of complex expressions. Throughout the following, we shall use the fact that it is frequently possible to determine α in such a manner that certain troublesome terms are eliminated from one side of an inequality, so that they will occur on the other side only.

LEMMA 2. *Assuming* R *to be real and positive,** *our identity may be written thus:*

$$\begin{aligned} |R^2 - \bar{u}_0 u|^2 - |R(u - u_0)|^2 &= (R^2 - |u_0|^2)(R^2 - |u|^2) \\ &= (R^2 - |u_0 u|)^2 - R^2(|u| - |u_0|)^2 \\ &= (R^2 + |u_0 u|)^2 - R^2(|u| + |u_0|)^2. \end{aligned}$$

LEMMA 3. *From the preceding lemma, the following three cases are obtained by comparing signs in the upper line and dividing by* $|R^2 - \bar{u}_0 u|^2$*:*

(*a*) *For* $|u_0| < R$, *the inequalities*

$$|u| \lesseqqgtr R \qquad \text{and} \qquad \left|\frac{R(u - u_0)}{R^2 - \bar{u}_0 u}\right| \lesseqqgtr 1$$

are equivalent; for the lower inequality signs it is necessary however that $R^2 \neq \bar{u}_0 u$.

(*b*) *For* $|u_0| > R$, *the same statement is true provided that the signs in*

---

* As everywhere in the following.

the second inequality are reversed; for the upper inequality signs it is necessary however that $R^2 \neq \bar{u}_0 u$.

(*c*) *For* $|u_0| = R$, *we always have*

$$\left|\frac{R(u - u_0)}{R^2 - \bar{u}_0 u}\right| = 1;$$

for $|u| = R$ *it is necessary however that* $u \neq u_0$ (or what is the same in this case, $R^2 \neq \bar{u}_0 u$).

Lemma 4. *When* $R^2 \neq \bar{u}_0 u$ *and* k *is a real, non-negative number satisfying the inequality* $k\,|u_0| < R$, *the inequalities*

$$\left|\frac{R(u - u_0)}{R^2 - \bar{u}_0 u}\right| \lesseqqgtr k \qquad \textit{and} \qquad \left|u - \frac{R^2(1 - k^2)}{R^2 - k^2\,|u_0|^2}u_0\right| \lesseqqgtr kR\left|\frac{R^2 - |u_0|^2}{R^2 - k^2\,|u_0|^2}\right|$$

are equivalent.

When $k\,|u_0| > R$, *this statement remains true when reversing the signs in the second inequality.*

Proof: The first inequality is equivalent to

$$|R(u - u_0)| \lesseqqgtr |k(R^2 - \bar{u}_0 u)|$$

and hence, by the α method, also to

$$|R(u - u_0) + \alpha k(R^2 - \bar{u}_0 u)| \lesseqqgtr |\bar{\alpha} R(u - u_0) + k(R^2 - \bar{u}_0 u)|$$

for $|\alpha| < 1$, while the signs are to be reversed for $|\alpha| > 1$. Making $\alpha = k\,|u_0|/R$, u is eliminated from the right side, and the lemma follows upon division by $|R^2 - k^2\,|u_0|^2| \cdot R^{-1}$, which is $\neq 0$ by our assumptions.

Note. For $k = 1$, lemma 3 (*a*) and (*b*) follows from this.

Lemma 5. *When* $|u_0| < R$ *and* $|u| < R$, *then*

$$\frac{R\,|\,|u| - |u_0|\,|}{R^2 - |u_0 u|} \leqq \left|\frac{R(u - u_0)}{R^2 - \bar{u}_0 u}\right| \leqq \frac{R(|u| + |u_0|)}{R^2 + |u_0 u|}.$$

The equality sign to the left will hold when and only when either one (*or both*) *of the numbers* u_0 *and* u *is zero, or* $\operatorname{sg}(u) = \operatorname{sg}(u_0)$, *and the equality sign to the right, when and only when either one* (*or both*) *of the numbers* u_0 *and* u *is zero or* $\operatorname{sg}(u) = -\operatorname{sg}(u_0)$.

Proof. By lemma 2, either side in the identities

$$|R^2 - \bar{u}_0 u|^2 - |R(u - u_0)|^2 = (R^2 - |u_0 u|)^2 - R^2(|u| - |u_0|)^2,$$

$$|R^2 - \bar{u}_0 u|^2 - |R(u - u_0)|^2 = (R^2 + |u_0 u|)^2 - R^2(|u| + |u_0|)^2$$

is positive. Dividing the upper and lower identity member by member by the inequalities

$$|R^2 - \bar{u}_0 u|^2 \geqq (R^2 - |u_0 u|)^2 \qquad \text{and} \qquad |R^2 - \bar{u}_0 u|^2 \leqq (R^2 + |u_0 u|)^2$$

respectively, inequalities are obtained which are equivalent to those to be proved. The statement regarding the equality signs follows from the fact that the equality signs in the inequalities used in the proof hold when and only when $\operatorname{sg}(\bar{u}_0 u) = 1$ or 0 and $\operatorname{sg}(\bar{u}_0 u) = -1$ or 0 respectively.

After these quite elementary algebraic lemmas* we now proceed to a function theoretic one; but first we shall introduce some definitions and notations which will be used everywhere in the following.

By x and x^* we denote complex variables satisfying the conditions $|x| < R$, $|x^*| < R$, unless the contrary is explicitly stated. Moreover, we denote by x_0 a particular value of x which may be chosen arbitrarily subject to the condition stated, viz. $|x_0| < R$. By $x_1, x_2, \cdots, x_n$ we denote other values *which will be defined in each individual case.*

With every x_ν $(\nu = 0, 1, 2, \cdots)$ we associate a real and non-negative κ_ν determined by

$$\kappa_\nu \equiv \kappa_\nu(x) = \left| \frac{R(x - x_\nu)}{R^2 - \bar{x}_\nu x} \right|.$$

It follows at once from lemma 3 (a) that $\kappa_0 < 1$ always, and that for other subscripts $\kappa_\nu \lesseqqgtr 1$ according as $|x_\nu| \lesseqqgtr R$; for the lower inequality sign it is necessary that $R^2 \neq \bar{x}_\nu x$. For brevity, we denote by $\kappa \equiv \kappa(x)$, a product of κ's with different subscripts, for instance $\kappa = \kappa_0 \kappa_1 \cdots \kappa_n$. When all of $|x_1|, |x_2|, \cdots, |x_n|$ are $< R$, this κ will satisfy the inequality $0 \leqq \kappa < 1$. When using the other variable x^*, we introduce the similar notations $\kappa_\nu^* \equiv \kappa(x_\nu^*)$ for $\nu = 0, 1, 2, \cdots$ and $\kappa^* \equiv \kappa(x^*)$; the constants x_0, x_1, x_2, $\cdots$ we do not change unless expressly stated. A particularly simple case occurs when κ reduces to κ_0, which equals $|x|/R$ under the special assumption $x_0 = 0$. Lemma 3 (a) leads to an important result. When all $|x_\nu| < R$, the corresponding $\kappa_\nu = 1$ for $|x| = R$, and consequently $\kappa \to 1$ as $|x| \to R$.

By $y \equiv y(x)$ we always denote an analytic function of x which is holomorphic for $|x| < R$; for $|x| = R$ we make no assumptions. Evidently $y(x)$ may be expressed as a power series with a radius of convergence $\geqq R$. By y^* we denote $y(x^*)$, and for brevity, we always write $y_0 = y(x_0)$ except for $x_0 = 0$, when the unabridged notation $y(0)$ is used to avoid ambiguity.

Apart from the algebraic lemmas already set down, we use in this paper only a few propositions from the most elementary part of the theory of functions, which are proved most simply, elegantly and advantageously (i.e., with the smallest amount of presupposed theory) by means of power series. Such propositions are: the quotient of two holomorphic functions is also a holomorphic function when the denominator vanishes nowhere for $| x | < R$; the maximum of the absolute value of a function holomorphic for $| x | \leqq r$ will occur only for an x-value (or values) such that $| x | = r$, unless the function is a constant, and so forth; and finally

---

* The reader will undoubtedly have noticed that lemmas 2–5 are homogeneous in form, so that it would have been sufficient to state them for $R = 1$. But this would bring no advantage whatever in the following, and the homogeneous form was chosen purposely.

LEMMA 6, GENERALIZATION OF SCHWARZ' LEMMA. *When the function* y *is holomorphic for* $| x | < R$ *and satisfies the inequality* $| y | < M$ *for* $| x | < R$, M *being real and positive,*[*] *and when* y *has zeros at* $x_1, x_2, \cdots, x_n$[†] *in the region considered, then*

$$| y | \leqq M\kappa, \qquad \text{where} \qquad \kappa = \kappa_1\kappa_2 \cdots \kappa_n.$$

This solution is complete (since this inequality leads a fortiori to $| y | < M$, and gives $y = 0$ for $\kappa = 0$ or $x = x_1, x_2, \cdots, x_n$).

PROOF. It follows from our assumptions that the function

$$\frac{y}{(x - x_1)(x - x_2) \cdots (x - x_n)}$$

and therefore also

$$u = y\frac{(R^2 - \bar{x}_1 x)(R^2 - \bar{x}_2 x) \cdots (R^2 - \bar{x}_n x)}{R(x - x_1) \cdot R(x - x_2) \cdots R(x - x_n)}$$

are holomorphic for $| x | < R$, and we have

$$| u | = \frac{| y |}{\kappa} < \frac{M}{\kappa}.$$

It has already been noted that $\kappa \to 1$ as $| x | \to R$, and consequently we may choose an $| x | = r < R$ so close to R that $r > | x_\nu |$ for $\nu = 1, 2, \cdots, n$ and that $\kappa > 1/(1 + \epsilon)$, where ϵ is any assigned positive number; we then have $| u | < M(1 + \epsilon)$ on the circumference $| x | = r$. By a theorem in the elementary theory of functions quoted above, the last inequality then also holds for all $| x | \leqq r$, and since ϵ is arbitrary, it is seen that for $| x | \leqq r < R$ we have $| u | \leqq M$ or $| y | \leqq M\kappa$, which completes the proof.

NOTE.[‡] The equality sign will obviously occur when $x = x_1$ or $x_2 \cdots$ or x_n, both sides being then equal to zero. Suppose however that $| y | = M\kappa$ for *one* value $x = x_0$ distinct from $x_1, x_2, \cdots, x_n$ and such that $| x_0 | < R$. Then $| u | = | y |/\kappa = M$ for $x = x_0$; on the other hand, we have proved that $| u | \leqq M$ for $| x | < R$. Since $| x_0 | < \frac{1}{2}(R + | x_0 |) < R$, u is holomorphic inside and on the circumference $| x | = \frac{1}{2}(R + | x_0 |)$, and $| u |$ takes its maximum value M at the point x_0 inside this circumference;

---

[*] As everywhere in the following.

[†] No x_ν may occur more times than indicated by the multiplicity of the corresponding zero of y. This condition is always understood in the following without being expressly stated in each case.

[‡] In the translation, this note has been changed so as to include the proof that the special rational function given below is the only one for which $| y | = M\kappa$.

by the function theoretic theorem quoted before, u is therefore a constant, and since $| u(x_0) | = M$, we have $u = \gamma M$, where γ is a constant such that $| \gamma | = 1$. Returning to the function y, it follows that

$$y = \gamma M \prod_{\nu=1}^{n} \frac{R(x - x_\nu)}{R^2 - \bar{x}_\nu x}.$$

Conversely, it is seen at once that this y is holomorphic for $| x | < R$, and that for *every* x where $| x | < R$, we have $| y | = M\kappa$ (and hence $| y | < M$). Since $\kappa \rightarrow 1$ as $| x | \rightarrow R$, we also have $| y | \rightarrow M$.

Before proceeding to apply the lemmas proved in this paragraph, we shall consider the last of them for a moment, writing it out at length:

When $y(x)$ is holomorphic and $| y | < M$ for $| x | < R$, and when $x_1, x_2, \cdots, x_n$ are some of the zeros of the function in the region considered, then

$$M \geqq \left| y(x) \frac{(R^2 - \bar{x}_1 x)(R^2 - \bar{x}_2 x) \cdots (R^2 - \bar{x}_n x)}{R(x - x_1) \cdot R(x - x_2) \cdots R(x - x_n)} \right|.$$

When moreover y is holomorphic in a larger region $| x | < R' (R' > R)$, the lemmas 3 (c), 3 (b) and 3 (a) allow us to include a fortiori among the x_ν occurring to the right any number of other x_ν situated quite arbitrarily in the extended region outside the original one, the corresponding factors $1/\kappa_\nu$ being $\leqq 1$ for $| x | < R$ (and since $| x | < R$, it is unnecessary to add the restriction $x \neq x_\nu$ for these new x_ν). Of course we are at liberty to let these x_ν coincide with any existing zeros of $y(x)$ in the new region.

This extension of lemma 6 includes, for $x = 0$, my theorem quoted above in § 1 no. 3, as well as the results of Carathéodory and Fejér in the paper quoted in [8].

Lemma 6 may also be extended in another direction. It is no essentially new problem to consider zeros of $y - \eta$ instead of zeros of y, η being an arbitrary constant subject however to the condition $| \eta | < M$. All that is necessary is to replace y by the function

$$\frac{M(y - \eta)}{M^2 - \bar{\eta} y}.$$

In fact, this function is holomorphic, our assumptions showing that the absolute value of the denominator is greater than zero, and by lemma 3 (a), the absolute value of the function is less than unity. We may therefore apply lemma 6, and thus find the following extension of the latter:

When y is a holomorphic function of x for $| x | < R$, and $| y | < M$ for $| x | < R$, and $x_1, x_2, \cdots, x_n$ are some of the zeros (interior to this region) of the function $y - \eta$, where η is a constant such that $| \eta | < M$, then*

$$M \, | y - \eta | \leqq | M^2 - \bar{\eta} y | \, \kappa,$$

where $\kappa = \kappa_1 \kappa_2 \cdots \kappa_n$.

---

* When $| \eta | \geqq M$, the region considered contains no zeros of $y - \eta$.

For $\eta = 0$, we again have lemma 6. What is said above about extending the latter will of course apply, mutatis mutandis, here also.

From $\kappa = 0$ we conclude that $y(x_\nu) = \eta$ for $\nu = 1, 2, \cdots, n$, and from the theorem it follows a fortiori that $M \mid y - \eta \mid < \mid M^2 - \bar{\eta} y \mid$ or $\mid y \mid < M$ (by lemma 3 (*a*)). Hence the theorem is complete.

For $x = 0$ we obtain the result of Landau ^{9}, p. 107; see above, p. 3.

§ 3. General theorems concerning functions bounded with respect to their absolute value.

In addition to the notations and definitions introduced in § 2 (p. 9) we shall assume throughout this paragraph that $\mid y \mid < M$ for $\mid x \mid < R$, unless otherwise stated.

Theorem 1. *When* x_0 *is fixed arbitrarily* within the region, and* $x_0, x_1, \cdots, x_n$ *denote some of the zeros of* $y - y_0$ *in the region (in particular, none of them except* x_0*), then we have the following three equivalent inequalities*

I. $$M \mid y - y_0 \mid \leqq \mid M^2 - \bar{y}_0 y \mid \kappa;$$

II. $$\mid y - y_0 \mid^2 \leqq \frac{(M^2 - \mid y_0 \mid^2)(M^2 - \mid y \mid^2)}{M^2} \frac{\kappa^2}{1 - \kappa^2};$$

III. $$\left| y - y_0 \frac{M^2(1 - \kappa^2)}{M^2 - \mid y_0 \mid^2 \kappa^2} \right| \leqq M \frac{M^2 - \mid y_0 \mid^2}{M^2 - \mid y_0 \mid^2 \kappa^2} \kappa;$$

where

$$\kappa = \kappa_0 \kappa_1 \cdots \kappa_n.$$

The theorem is complete, i.e., all the assumptions follow again from either one of the inequalities I, II *or* III, *provided that there exists an* x_0 *for which* $\mid y_0 \mid < M$.

Proof. On account of the assumptions, the function $y - y_0$ has zeros for $x = x_0, x_1, \cdots, x_n$, and I is therefore an immediate consequence of lemma 6 in the extended form given at the end of § 2. Supposing conversely that inequality I is true, then $y = y_0$ for $\kappa = 0$, and from I it follows a fortiori that $M \mid y - y_0 \mid < \mid M^2 - \bar{y}_0 y \mid$. When there exists an x_0 for which $\mid y_0 \mid < M$, then we also have $\mid y \mid < M$ by lemma 3 (*a*), and consequently I is complete.

Squaring both sides of I and using the identity in lemma 2, it follows that equivalently

$$M^2 \mid y - y_0 \mid^2 \leqq \mid M^2 - \bar{y}_0 y \mid^2 \kappa^2$$
$$= [M^2 \mid y - y_0 \mid^2 + (M^2 - \mid y_0 \mid^2)(M^2 - \mid y \mid^2)] \kappa^2,$$

and this is equivalent to II.

---

* In the applications it is of course most advantageous to choose x_0 so that $y_0 \equiv y(x_0)$ has a convenient value.

Applying lemma 4 to I (writing y for u, y_0 for u_0, M for R and κ for k) we find, since $|y_0|\kappa < |y_0| < M$, that I and III are equivalent. Thus theorem 1 is proved in its entirety.

NOTE 1. I and II are also true under the assumptions $|y| > M$, $|y_0| > M$, as is seen at once upon replacing y by $1/y$ and y_0 by $1/y_0$; this also follows from lemma 3 (*b*).

NOTE 2.* The equality sign always holds in I, II and III for the particular values $x = x_0$ or $x_1 \cdots$ or x_n. If it holds for an x distinct from these, it follows from the note to lemma 6 that y is of the form

$$y = M \frac{y_0 + \gamma M R^{n+1} \prod_{\nu=0}^{n} \frac{x - x_\nu}{R^2 - \bar{x}_\nu x}}{M + \gamma \bar{y}_0 R^{n+1} \prod_{\nu=0}^{n} \frac{x - x_\nu}{R^2 - \bar{x}_\nu x}},$$

where $|y_0| < M$ and $|\gamma| = 1$, and conversely, the equality sign holds in I, II and III for all x such that $|x| < R$ when y is of the form indicated.

NOTE 3. For $y_0 = 0$, I, II and III reduce to lemma 6.

THEOREM 2. *Under the same assumptions as in theorem I we have*

$$(a) \qquad |y - y_0| \leqq \frac{M^2 - |y_0|^2}{M - |y_0|\kappa}\kappa;$$

$$(b) \qquad |y - y_0| \leqq (M + |y_0|\kappa)\kappa;$$

$$(c) \qquad |y - y_0| \leqq \sqrt{M^2 - |y_0|^2}\frac{\kappa}{\sqrt{1-\kappa^2}}.$$

PROOF. (*a*) follows from theorem 1, I upon replacing

$$|M^2 - \bar{y}_0 y| = |M^2 - |y_0|^2 - \bar{y}_0(y - y_0)|$$

a fortiori by $M^2 - |y_0|^2 + |y_0||y - y_0|$. (*b*) follows a fortiori from (*a*), and (*c*) a fortiori from theorem 1, II.

NOTE 1. When an x_0 exists for which $y_0 = 0$, (*a*) and (*b*) are evidently complete.

NOTE 2. When

$$\kappa = \kappa_0 = \frac{R|x - x_0|}{R^2 - \bar{x}_0 x},$$

it is easy to derive from (*a*), (*b*) or (*c*) upper bounds for the absolute value of the difference quotient $(y - y_0)/(x - x_0)$. For $x \to x_0$, the first of these gives Lindelöf's formula (upon replacing x_0 by x)

$$\left|\frac{dy}{dx}\right| \leqq \frac{R(M^2 - |y|^2)}{M(R^2 - |x|^2)}.\dagger$$

---

* This note has been changed to correspond to the changes in the note to lemma 6.

† See above, p. 4.

We shall return to this in the following in more general (and better) forms.

THEOREM 3. *Under the same assumptions as in theorem* 1 *we have, η being a constant,*

$$\eta - y_0\frac{M^2(1-\kappa^2)}{M^2-|y_0|^2\kappa^2}\Bigg| - M\frac{M^2-|y_0|^2}{M^2-|y_0|^2\kappa^2}\kappa \leqq |y-\eta|$$

$$\leqq \left|\eta - y_0\frac{M^2(1-\kappa^2)}{M^2-|y_0|^2\kappa^2}\right| + M\frac{M^2-|y_0|^2}{M^2-|y_0|^2\kappa^2}\kappa.$$

THE PROOF of the two inequalities follows from theorem 1, III upon replacing the left side by

$$\left|\eta - y_0\frac{M^2(1-\kappa^2)}{M^2-|y_0|^2\kappa^2}\right| - |y-\eta|$$

and

$$|y-\eta| - \left|\eta - y_0\frac{M^2(1-\kappa^2)}{M^2-|y_0|^2\kappa^2}\right|$$

respectively.

NOTE 1. For $\kappa = 0$, both inequalities reduce to the identity

$$|y(x_\nu) - \eta| = |y_0 - \eta|.$$

NOTE 2. For $\eta = y_0$, the inequality to the right reduces to theorem 2, (*a*).

COROLLARY 1. When $\eta = 0$, theorem 3 takes the following form (after a simple reduction to the right and left)

$$M\frac{|y_0| - M\kappa}{M - |y_0|\kappa} \leqq |y| \leqq M\frac{|y_0| + M\kappa}{M + |y_0|\kappa}.$$

(Before going further, we shall prove this corollary in a different manner by the inequality to the left in lemma 5. From our assumptions, it follows that

$$\left|\frac{M(y-y_0)}{M^2-\bar{y}_0 y}\right| \leqq \kappa,$$

or a fortiori

$$\frac{M(|y_0|-|y|)}{M^2-|y_0|\,|y|} \leqq \kappa \qquad \text{and} \qquad \frac{M(|y|-|y_0|)}{M^2-|y_0|\,|y|} \leqq \kappa,$$

and from these, the inequalities to be proved follow equivalently by solving in respect to $|y|$.)

Moreover, it may be noted that when there exists an x_0 such that $y_0 = 0$, the inequality to the right reduces (for this value of x_0) to lemma 6, the generalized lemma of Schwarz, and is then complete. Besides, we always have

$$M\frac{|y_0| + M\kappa}{M + |y_0|\kappa} = M\left(1 - \frac{(M - |y_0|)(1 - \kappa)}{M + |y_0|\kappa}\right) < M$$

when

$$|y_0| < M.$$

A noteworthy special case is $\kappa = \kappa_0$,* or

$$M\frac{|y_0|\,|R^2 - \bar{x}_0 x| - MR\,|x - x_0|}{M\,|R^2 - \bar{x}_0 x| - R\,|y_0(x - x_0)|} \leqq |y|$$
$$\leqq M\frac{|y_0|\,|R^2 - \bar{x}_0 x| + MR\,|x - x_0|}{M\,|R^2 - \bar{x}_0 x| + R\,|y_0(x - x_0)|},$$

which reduces, for $x_0 = 0$ (and $y_0 = y(0)$), to two of Lindelöf's formulas (l.c. [11]), formula (4) p. 12,† and formula (1), p. 11, compare above, p. 4).

From corollary 1 we deduce

COROLLARY 2.

$$M^2\frac{M^2 - |y_0|^2}{M^2 - |y_0|^2\kappa^2}(1 - \kappa)^2 \leqq M^2 - |y|^2 \leqq M^2\frac{M^2 - |y_0|^2}{M^2 - |y_0|^2\kappa^2}(1 + \kappa)^2.$$

PROOF. Corollary 1 may be written

$$M\frac{M + |y_0|}{M - |y_0|\kappa}(1 - \kappa) \leqq M + |y| \leqq M\frac{M + |y_0|}{M + |y_0|\kappa}(1 + \kappa)$$

and

$$M\frac{M - |y_0|}{M + |y_0|\kappa}(1 - \kappa) \leqq M - |y| \leqq M\frac{M - |y_0|}{M - |y_0|\kappa}(1 + \kappa),$$

whence corollary 2 follows upon multiplication member by member. In the following, we shall have occasion to use the inequality to the right in this corollary; but it should be stated explicitly that this cannot be used to advantage unless κ has such a value that the right-hand member is $< M^2$; otherwise the inequality is trivial, and it is better simply to use M^2. Since we may replace the right-hand member a fortiori by

$$M^2\frac{M^2 - |y_0|^2}{M^2(1 - \kappa^2)}(1 + \kappa)^2 = (M^2 - |y_0|^2)\frac{1 + \kappa}{1 - \kappa},$$

it is seen that the formula is advantageous when

$$\kappa < \frac{|y_0|^2}{2M^2 - |y_0|^2},$$

or at any rate for

$$\kappa < \tfrac{1}{2}\frac{|y_0|^2}{M^2}.$$

---

* Where consequently no assumption is made regarding the existence of the zeros x_1, x_2, $\cdots$.

† From which however a superfluous restriction is removed.

COROLLARY 3. When $0 < M' < |y| < M$, then

$$M'\frac{|y_0| + M'\kappa}{M' + |y_0|\kappa} \leqq |y| \leqq M\frac{|y_0| + M\kappa}{M + |y_0|\kappa}.$$

PROOF. The inequality to the right has already been proved in corollary 1, and the one to the left follows by applying the inequality to the right to the function $1/y$ and replacing y_0 and M by $1/y_0$ and $1/M'$.

THEOREM 4. *Under the same assumptions as in theorem* 1, *and making* η $(\neq y_0)$ *a constant less than M in absolute value,*[*] *then*

$$y \neq \eta \qquad \textit{for} \qquad \kappa = \kappa_0\kappa_1\cdots\kappa_n < \left|\frac{M(y_0 - \eta)}{M^2 - \bar{\eta}y_0}\right|.^{\dagger}$$

PROOF. $y \neq \eta$ when the first member in theorem 3 is positive, or when

$$\left|\eta - y_0\frac{M^2(1 - \kappa^2)}{M^2 - |y_0|^2\kappa^2}\right| > M\frac{M^2 - |y_0|^2}{M^2 - |y_0|^2\kappa^2}\kappa.$$

By lemma 4 (writing κ for k, M for R, η for u and y_0 for u_0; it follows from our assumptions that $M^2 \neq \bar{y}_0\eta$ and $\kappa|y_0| < M$) this inequality is equivalent to

$$\left|\frac{M(\eta - y_0)}{M^2 - \bar{y}_0\eta}\right| = \left|\frac{M(y_0 - \eta)}{M^2 - \bar{\eta}y_0}\right| > \kappa,$$

which proves the theorem.

Another proof is obtained by applying the left side of corollary 1, theorem 3, to the function

$$\frac{M(y - \eta)}{M^2 - \bar{\eta}y},$$

which is less than unity in absolute value (as shown at the end of § 2), and which moreover takes the value

$$\frac{M(y_0 - \eta)}{M^2 - \bar{\eta}y_0}$$

for $x = x_\nu$ $(\nu = 0, 1, 2, \cdots, n)$, whence

$$\left|\frac{M(y - \eta)}{M^2 - \bar{\eta}y}\right| \geqq \frac{\left|\dfrac{M(y_0 - \eta)}{M^2 - \bar{\eta}y_0}\right| - \kappa}{1 - \left|\dfrac{M(y_0 - \eta)}{M^2 - \bar{\eta}y_0}\right|\kappa}.$$

---

[*] For $|\eta| \geqq M$, we have $y \neq \eta$ in the entire region $|x| < R$.

[†] By lemma 2, this condition may also be written

$$\kappa^2 < \frac{M^2|y_0 - \eta|^2}{M^2|y_0 - \eta|^2 + (M^2 - |\eta|^2)(M^2 - |y_0|^2)}.$$

Note. It is apparent at once that the more factors it is possible to put into κ on the left in the inequality in theorem 4, the larger will be the region it determines for x. The significance of the theorem is thus greatest when we know as many of the zeros of $y - y_0$ as possible; but we may always make $\kappa = \kappa_0$ and obtain

Corollary 1.

$$y \neq \eta \qquad \text{for} \qquad \kappa_0 = \left| \frac{R(x - x_0)}{R^2 - \bar{x}_0 x} \right| < \left| \frac{M(y_0 - \eta)}{M^2 - \bar{\eta} y_0} \right| .$$

We note the particular case $x_0 = 0$ (and $y_0 = y(0)$), which gives the result due to Landau (l.c.[9], p. 105) quoted p. 3 above.

Another noteworthy special case is $\eta = 0$, which gives

$$y \neq 0 \qquad \text{for} \qquad \left| \frac{R(x - x_0)}{R^2 - \bar{x}_0 x} \right| < \frac{|\, y_0 \,|}{M} .$$

Hence we obtain a fortiori, replacing

$$|\, R^2 - \bar{x}_0 x \,| = |\, R^2 - |\, x_0 \,|^2 - \bar{x}_0 (x - x_0) \,|$$

by $R^2 - |\, x_0 \,|^2 - |\, x_0 \,| \, |\, x - x_0 \,|$, the result of Lindelöf (l.c.[11], formula (5), p. 12):

$$y \neq 0 \qquad \text{for} \qquad |\, x - x_0 \,| < \frac{R^2 - |\, x_0 \,|^2}{RM + |\, x_0 y_0 \,|} \, |\, y_0 \,| .$$

Corollary 2. The preceding corollary may be transformed equivalently into

$$y \neq \eta \qquad \text{for} \qquad \left| x - x_0 \frac{R^2(|\, M^2 - \bar{\eta} y_0 \,|^2 - M^2 \,|\, y_0 - \eta \,|^2)}{R^2 \,|\, M^2 - \bar{\eta} y_0 \,|^2 - M^2 \,|\, x_0(y_0 - \eta) \,|^2} \right| < \frac{RM(R^2 - |\, x_0 \,|^2) \,|\, M^2 - \bar{\eta} y_0 \,| \, |\, y_0 - \eta \,|}{R^2 \,|\, M^2 - \bar{\eta} y_0 \,|^2 - M^2 \,|\, x_0(y_0 - \eta) \,|^2} ,$$

and in particular

$$y \neq 0 \qquad \text{for} \qquad \left| x - x_0 \frac{R^2(M^2 - |\, y_0 \,|^2)}{R^2 M^2 - |\, x_0 y_0 \,|^2} \right| < \frac{RM(R^2 - |\, x_0 \,|^2) \,|\, y_0 \,|}{R^2 M^2 - |\, x_0 y_0 \,|^2} .$$

The last region is larger in every direction than the one due to Lindelöf which we have just quoted.

The proof follows by using lemma 4 for the transformation of corollary 1 (writing $|\, [M(y_0 - \eta)]/[M^2 - \bar{\eta} y_0] \,|$ for k, x for u, and x_0 for u_0; we then have

$$|\, x_0 \,| \left| \frac{M(y_0 - \eta)}{M^2 - \bar{\eta} y_0} \right| < |\, x_0 \,| < R).$$

Using the identity in lemma 2, we also have

$$y \neq \eta \text{ for } \left| x - x_0 \frac{R^2(M^2 - |\eta|^2)(M^2 - |y_0|^2)}{R^2(M^2 - |\eta|^2)(M^2 - |y_0|^2) + M^2 |y_0 - \eta|^2 (R^2 - |x_0|^2)} \right|$$
$$< \frac{RM(R^2 - |x_0|^2)\,|M^2 - \bar{\eta} y_0|\,|y_0 - \eta|}{R^2(M^2 - |\eta|^2)(M^2 - |y_0|^2) + M^2 |y_0 - \eta|^2 (R^2 - |x_0|^2)}$$

THEOREM 5, CONCERNING THE DERIVATIVE OF y. *Under the same assumptions as in theorem* 1, *we have*

$$\left| \frac{dy}{dx} \right| \leqq \frac{R(M^2 - |y|^2)}{M(R^2 - |x|^2)} \leqq \frac{RM}{R^2 - |x|^2} \frac{M^2 - |y_0|^2}{M^2 - |y_0|^2 \kappa^2} (1 + \kappa)^2 *$$

PROOF. As we have noted before, the first inequality, Lindelöf's formula, is an immediate consequence of theorem 1, I or II, making $\kappa = \kappa_0$ and $x_0 \rightarrow x$. The second inequality follows from theorem 3, corollary 2, inequality to the right.

NOTE 1. The inequality obtained a fortiori by leaving out the middle member may be said to be more extensive in a certain sense than Lindelöf's formula, since the latter results from the former for $x_0 = x$. Moreover, the third member contains not y but y_0, which may be regarded as known.

NOTE 2.† To find when the equality sign holds in this theorem, let us assume that for $x = x_0$ and $y = y_0$ ($|x_0| < R$, $|y_0| < M$) we have

$$\left| \frac{dy}{dx} \right|_0 = \frac{R(M^2 - |y_0|^2)}{M(R^2 - |x_0|^2)}.$$

The function

$$u = \frac{M(y - y_0)}{M^2 - \bar{y}_0 y} \frac{R^2 - \bar{x}_0 x}{R(x - x_0)}$$

is holomorphic for $|x| < R$, and making $\kappa = \kappa_0$ in theorem 1, I, we find $|u| \leqq 1$ for $|x| < R$. Moreover,

$$u(x_0) = \lim_{x \rightarrow x_0} u = \frac{M(R^2 - |x_0|^2)}{R(M^2 - |y_0|^2)} \lim_{x \rightarrow x_0} \frac{y - y_0}{x - x_0} = \frac{M(R^2 - |x_0|^2)}{R(M^2 - |y_0|^2)} \left(\frac{dy}{dx} \right)_0,$$

and using the value of $|dy/dx|_0$, it is seen that $|u(x_0)| = 1$. Since $|x_0| < \frac{1}{2}(R + |x_0|) < R$, u is holomorphic inside and on the circumference

---

* Instead of the first inequality we may write a fortiori

$$\left| \frac{dy}{dx} \right| \leqq \frac{RM}{R^2 - |x|^2}.$$

When it is also known that $|y| > M' > 0$ in the region considered, we have the stronger inequality

$$\left| \frac{dy}{dx} \right| < \frac{R(M^2 - M'^2)}{M(R^2 - |x|^2)}.$$

† This note has been changed in the translation to correspond to the changes in the note to lemma 6.

$|x| = \frac{1}{2}(R + |x_0|)$, and $|u|$ takes its maximum value unity at the point x_0 inside this circumference; by the function theoretic theorem quoted immediately before lemma 6, u is therefore a constant $= \gamma$, where $|\gamma| = 1$. Substituting $u = \gamma$ in the equation defining u and solving for y, we obtain

$$y = M\frac{y_0(R^2 - \bar{x}_0 x) + \gamma M R(x - x_0)}{M(R^2 - \bar{x}_0 x) + \gamma \bar{y}_0 R(x - x_0)}.$$

Retracing the steps of the proof, it is seen conversely that for any y of this form, where $|x_0| < R$, $|y_0| < M$, $|\gamma| = 1$, the equality sign holds to the left in our theorem when $x = x_0$.

NOTE 3. We have observed before (see above, p. 4) that Lindelöf's formula contains for $x = 0$ an earlier formula due to Landau which, writing y as a power series in x

$$y(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

with radius of convergence $\geqq R$, may be expressed thus:

$$RM\,|c_1| \leqq M^2 - |c_0|^2.^*$$

It is easy to give a more general form to this result. Let p and q be integers, $q > p \geqq 0$ and let θ be a primitive qth root of unity; then

$$\frac{1}{q}\sum_{\nu=1}^{q} \theta^{-p\nu} y(x\theta^{\nu}) = x^p(c_p + c_{p+q}x^q + c_{p+2q}x^{2q} + \cdots)$$

where the expression to the left is less than M in absolute value, and the series in parentheses to the right, considered as a power series in x^q, converges at least for $|x^q| < R^q$. Considering all values of x for which $|x| = r < R$, the series in parentheses will be $< M/r^p$ in absolute value for all these x, and therefore also for all $|x| < r$. But r may be taken as close to R as we please, and the absolute value of the series will therefore be $\leqq M/R^p$. Hence, by Landau's theorem

$$R^q \frac{M}{R^p}\,|c_{p+q}| \leqq \frac{M^2}{R^{2p}} - |c_p|^2,$$

or

$$R^{p+q}M\,|c_{p+q}| \leqq M^2 - R^{2p}\,|c_p|^2,$$

---

* That Landau (l. c.[2]) proves this formula for $R = 1$, $M = 1$, implies no restriction. Incidentally, Lindelöf's more general formula may be derived from Landau's by a linear substitution. Writing

$$y(x) = f\left(\frac{R^2(x + x_0)}{R^2 + \bar{x}_0 x}\right),$$

then, by lemma 3*a*, $|f(x)| < M$ for $|x| < R$, and $f(x)$ is seen to be holomorphic in this region. The formula $RM\,|y'(0)| \leqq M^2 - |y(0)|^2$ becomes

$$\frac{M}{R}\,|f'(x_0)|\,(R^2 - |x_0|^2) \leqq M^2 - |f(x_0)|^2,$$

which gives Lindelöf's formula upon replacing $f(x)$ by $y(x)$ and x_0 by x.

which is the relation we desired to prove.* In the particular case $p = 0$, we obtain the generalization of Landau's theorem due to Wiener and quoted on p. 4.

THEOREM 6, CONCERNING THE DIFFERENCE QUOTIENT OF y. *Under the same assumptions as in theorem 1, we have for $x \neq x^*$*

$$\left|\frac{y - y^*}{x - x^*}\right|^2 \leqq \frac{R^2(M^2 - |y|^2)(M^2 - |y^*|^2)}{M^2(R^2 - |x|^2)(R^2 - |x^*|^2)}\,\dagger$$

where x^* is a new independent variable, and $y^* \equiv y(x^*)$. When, conversely, this inequality is satisfied, and there exists a certain value of x for which $|y| < M$, then $|y| < M$ for all x.

PROOF. In theorem 1, II, we make

$$\kappa = \kappa_0 = \frac{R|x - x_0|}{|R^2 - \bar{x}_0 x|},$$

replace x_0 by x^* and obtain

$$\left|\frac{y - y^*}{x - x^*}\right|^2 \leqq \frac{R^2(M^2 - |y|^2)(M^2 - |y^*|^2)}{M^2(|R^2 - \bar{x}^* x|^2 - R^2|x - x^*|^2)};$$

applying the identity in lemma 2, this becomes the desired inequality. When there exists a certain value of x^* for which $|y^*| < M$, we find conversely by theorem 1 that the inequality of theorem 6 is equivalent to

$$\left|\frac{M(y - y^*)}{M^2 - \bar{y}^* y}\right| \leqq \left|\frac{R(x - x^*)}{R^2 - \bar{x}^* x}\right|,$$

whence a fortiori

$$\left|\frac{M(y - y^*)}{M^2 - \bar{y}^* y}\right| < 1,$$

and this is equivalent to $|y| < M$ by lemma 3 (a). Thus the proof of theorem 6 is complete.

NOTE 1. By note 1 to theorem 1 it is readily seen that when there exists a certain value of x^* for which $|y^*| > M$, then the inequality in theorem 6 will also imply that $|y| > M$ for all values of x in the region considered.

---

* It is easy to generalize this result considerably. But this would fall beyond the scope of the present paper, and must be reserved for another occasion.

† Hence a fortiori the simple formula

$$\left|\frac{y - y^*}{x - x^*}\right| \leqq \frac{RM}{\sqrt{(R^2 - |x|^2)(R^2 - |x^*|^2)}}.$$

When it is also known that $|y| > M' > 0$ in the region considered, we have the more accurate formula

$$\left|\frac{y - y^*}{x - x^*}\right| < \frac{R(M^2 - M'^2)}{M\sqrt{(R^2 - |x|^2)(R^2 - |x^*|^2)}}.$$

NOTE 2. When it is desirable to avoid y or y^* to the right in the inequality in theorem 6, we may eliminate either or both of these functions as before, so that there occurs instead a y_0 which may be preferable. To this purpose, we replace a fortiori

$$M^2 - |y|^2 \quad \text{by} \quad M^2\frac{M^2 - |y_0|^2}{M^2 - |y_0|^2\kappa^2}(1+\kappa)^2,$$

(theorem 3, corollary 2), and

$$M^2 - |y^*|^2 \quad \text{by} \quad M^2\frac{M^2 - |y_0|^2}{M^2 - |y_0|^2\kappa^{*2}}(1+\kappa^*)^2,$$

where

$$\kappa^* \equiv \kappa(x^*).$$

Evidently this may be varied in many ways, for instance by replacing κ by κ_0 and κ^* by the κ_0^* obtained by changing x_0 into x_0^*, which necessitates replacing y_0 by $y(x_0^*)$ in the second of the expressions above. The formula thus obtained—without assumptions regarding $x_1, x_2, \cdots$—is very general. For $x_0 = x$ and $x_0^* = x^*$ this of course reduces to theorem 6.

The above will undoubtedly be sufficient to show how we may proceed in other cases when an upper or lower bound of $|y|$ is given.

A last general remark may be appropriate. Let us assume that under the assumptions in theorem 1 (viz., $|y| < M$ for $|x| < R$, $|x_0| < R$ and $x_1, x_2, \cdots, x_n$ some of the zeros of $y - y_0$ in the given region) we have found an expression (non-analytic) in $y, y_0, M, x, x_0, x_1, \cdots, x_n, R$ satisfying the condition

$$\text{(I)} \qquad U\left(\frac{y}{M}, \frac{y_0}{M}\right) \geqq 0.$$

Since, for η constant and $|\eta| < M$, $|y| < M$ is equivalent to

$$\left|\frac{M(y-\eta)}{M^2 - \bar{\eta}y}\right| < 1,$$

and $M(y-\eta)/(M^2 - \bar{\eta}y)$ takes the value $M(y_0 - \eta)/(M^2 - \bar{\eta}y_0)$ for $x = x_0, x_1, \cdots, x_n$, the more general formula

$$\text{(II)} \qquad U\left(\frac{M(y-\eta)}{M^2 - \bar{\eta}y}, \frac{M(y_0-\eta)}{M^2 - \bar{\eta}y_0}\right) \geqq 0$$

follows from (I), which in turn results from (II) by making $\eta = 0$. Again, making $\eta = y_0$, we find the simplified formula

$$U\left(\frac{M(y-y_0)}{M^2 - \bar{y}_0 y}, 0\right) \geqq 0.$$

This may be varied in many ways. As we have already done in particular cases, η may be taken as a function of another independent variable $x,^*$ particularly when $x_1, x_2, \cdots, x_n$ are disregarded.

Although the theorems in the next paragraph concerning a bounded $\Re(y)$ become in fact more or less direct corollaries to the developments of the present paragraph, when using the formulas for the connection between absolute value, real part, etc., given at the beginning of § 2, the details are of sufficient interest to justify us in dealing at some length with theorems of this kind.

§ 4. General theorems concerning functions bounded with respect to their real part.

In addition to the notations and definitions introduced in § 2, we shall assume throughout this paragraph that, unless otherwise stated, $\Re(y) < A$ for $|x| < R$, where A denotes a real number.

Theorem 1. *When x_0 is fixed arbitrarily within the region $|x| < R$, and $x_0, x_1, \cdots, x_n$ denote some of the zeros of the function $y - y_0$ in this region (in particular, none of them except x_0), then we have the following three equivalent inequalities:*

I. $$|y - y_0| \leqq |y - y_0 - 2(A - \Re y_0)| \kappa;$$

II. $$|y - y_0|^2 \leqq 4(A - \Re y_0)(A - \Re y) \frac{\kappa^2}{1 - \kappa^2};^*$$

III. $$\left| y - y_0 + 2(A - \Re y_0) \frac{\kappa^2}{1 - \kappa^2} \right| \leqq 2(A - \Re y_0) \frac{\kappa}{1 - \kappa^2},$$

where $\kappa = \kappa_0 \kappa_1 \cdots \kappa_n$.

The theorem is complete, i.e., all the assumptions follow again from either one of the inequalities I, II *or* III, *provided that there exists an x_0 for which $\Re(y_0) < A$.*

Proof. We write for brevity (as in all the following proofs)

$$a = A - \Re(y), \qquad a_0 = A - \Re(y_0).$$

Then $a > 0$, $a_0 > 0$ and $\Re(y - y_0) = a_0 - a < a_0$, which is equivalent to $4a_0\Re(y - y_0) < 4a_0^2$ and to

$$|y - y_0|^2 < |y - y_0|^2 - 4a_0\Re(y - y_0) + 4a_0^2 = |y - y_0 - 2a_0|^2$$

by the first identity in § 2. Consequently, since $\Re(y - y_0 - 2a_0) < -a_0 < 0$,

$$\frac{y - y_0}{y - y_0 - 2a_0}$$

is a function of x holomorphic in the given region; moreover it is < 1 in

---

* Or

$$\kappa^2 \geqq \frac{|y - y_0|^2}{|y - y_0|^2 + 4(A - \Re y_0)(A - \Re y)}.$$

absolute value and has the zeros x_0 and $x_1, x_2, \cdots, x_n$, if the latter are assumed to exist. Lemma 6 therefore gives

$$\left| \frac{y - y_0}{y - y_0 - 2a_0} \right| \leqq \kappa, \qquad \kappa = \kappa_0 \kappa_1 \cdots \kappa_n,$$

which is equivalent to I. Conversely, when we assume that inequality I is true and that there exists an x_0 for which $\Re(y_0) < A$, we have $y = y_0$ for $\kappa = 0$, and also, a fortiori, $| y - y_0 | < | y - y_0 - 2a_0 |$ which we have just shown to be equivalent to $\Re(y - y_0) < a_0$ or $\Re(y) < A$; the inequality I is therefore complete.

Squaring I, we have the equivalent inequality

$$| y - y_0 |^2 \leqq (| y - y_0 |^2 - 4a_0 \Re(y - y_0) + 4a_0^2)\kappa^2 = | y - y_0 |^2 \kappa^2 + 4a_0 a \kappa^2$$

which is equivalent to II.

III may also be proved by squaring, but we prefer the α method. For $| \alpha | < 1$, inequality I is equivalent to

$$| y - y_0 + \alpha(y - y_0 - 2a_0)\kappa | \leqq | \bar{\alpha}(y - y_0) + (y - y_0 - 2a_0)\kappa |$$

which, for $\alpha = \bar{\alpha} = - \kappa$ (which satisfies the condition $| \alpha | < 1$), is equivalent to

$$| (y - y_0)(1 - \kappa^2) + 2a_0\kappa^2 | \leqq 2a_0\kappa,$$

which again is equivalent to III. Thus theorem I is completely proved.

NOTE 1. When we replace y by $- y$ and A by $- A$ in theorem 1, it is seen that I and II remain unchanged; these two inequalities are therefore also true under the assumption $\Re(y) > A$ for $| x | < R$.

NOTE 2.* The equality sign in I, II and III always holds for $\kappa = 0$ or $x = x_0, x_1, \cdots, x_n$. If it holds for an x distinct from these, it follows from the note to lemma 6 that

$$\frac{y - y_0}{y - y_0 - 2a_0} = \gamma \prod_{\nu=0}^{n} \frac{R(x - x_\nu)}{R^2 - \bar{x}_\nu x}$$

where γ is a constant such that $| \gamma | = 1$, or

$$y = y_0 - 2(A - \Re y_0) \frac{\gamma R^{n+1} \prod_{\nu=0}^{n} \frac{x - x_\nu}{R^2 - \bar{x}_\nu x}}{1 - \gamma R^{n+1} \prod_{\nu=0}^{n} \frac{x - x_\nu}{R^2 - \bar{x}_\nu x}},$$

and conversely, the equality sign in I, II and III holds for all x such that $| x | < R$ when y is of the form indicated.

---

* This note has been changed to correspond to the changes in the note to lemma 6.

Theorem 2. *Under the same assumptions as in theorem* 1, *we have*

$$| y - y_0 | \leqq 2(A - \Re y_0) \frac{\kappa}{1 - \kappa}.$$

Proof from theorem 1, I upon replacing the right-hand member a fortiori by $| y - y_0 | \kappa + 2a_0\kappa$.*

Note 1. From the inequality in this theorem we may again derive some of the assumptions, viz., $y = y_0$ for $x = x_1, x_2, \cdots, x_n$.

Note 2. For $\kappa = \kappa_0$ we have

$$| y - y_0 | \leqq 2(A - \Re y_0) \frac{R \mid x - x_0 \mid}{\mid R^2 - \bar{x}_0 x \mid - R \mid x - x_0 \mid},$$

and making $x_0 = 0$ in this inequality, we obtain a theorem due to Lindelöf (l.c.[11], p. 15)

$$| y - y(0) | \leqq 2(A - \Re y(0)) \frac{\mid x \mid}{R - \mid x \mid},$$

whence Carathéodory's theorem (see above, p. 3) follows a fortiori.†

Assuming $x \neq x_0$, dividing the first formula above by $|x - x_0|$ and letting $x_0 \rightarrow x$, we find another theorem due to Lindelöf (l.c.)

$$\left| \frac{dy}{dx} \right| \leqq 2R \frac{A - \Re(y)}{R^2 - \mid x \mid^2},$$

to which we shall return later.

Theorem 3. *Under the same assumptions as in theorem* 1, *and* η *being a constant, we have*

$$| y - \eta | \geqq \left| \eta - y_0 + 2(A - \Re y_0) \frac{\kappa^2}{1 - \kappa^2} \right| - 2(A - \Re y_0) \frac{\kappa}{1 - \kappa^2},$$

$$| y - \eta | \leqq \left| \eta - y_0 + 2(A - \Re y_0) \frac{\kappa^2}{1 - \kappa^2} \right| + 2(A - \Re y_0) \frac{\kappa}{1 - \kappa^2}.$$

The proof follows a fortiori from theorem 1, III upon replacing the left member of the inequality by

$$\left| \eta - y_0 + 2a_0 \frac{\kappa^2}{1 - \kappa^2} \right| - | y - \eta |$$

---

* In a particular case, viz., when x is known to have such a value that $\Re(y) \geqq \Re(y_0)$, theorem 1, II gives the stronger inequality

$$| y - y_0 | \leqq 2(A - \Re y_0) \frac{\kappa}{\sqrt{1 - \kappa^2}}.$$

† From theorem 2, it follows a fortiori that in the general case

$$| y | \leqq | y_0 | + 2(A - \Re y_0) \frac{\kappa}{1 - \kappa} \leqq | \Im y_0 | + 2A \frac{\kappa}{1 - \kappa} + | \Re y_0 | \frac{1 + \kappa}{1 - \kappa},$$

a direct generalization of Carathéodory's theorem.

and

$$|y - \eta| - \left|\eta - y_0 + 2a_0\frac{\kappa^2}{1 - \kappa^2}\right|$$

respectively.

NOTE 1. For $\kappa = 0$, both inequalities reduce to identities.

NOTE 2. For $\eta = y_0$, the lower inequality reduces to theorem 2.

NOTE 3. As a special case of theorem 3 we find for $\eta = 0$, $\kappa = \kappa_0$ and using the identity in lemma 2:

$$|y| \geqq \left|y_0 - 2(A - \mathfrak{R}y_0)\frac{R^2\,|x - x_0|^2}{(R^2 - |x_0|^2)(R^2 - |x|^2)}\right| - 2(A - \mathfrak{R}y_0)\frac{R\,|x - x_0|\,|R^2 - \bar{x}_0x|}{(R^2 - |x_0|^2)(R^2 - |x|^2)}$$

and

$$|y| \leqq \left|y_0 - 2(A - \mathfrak{R}y_0)\frac{R^2\,|x - x_0|^2}{(R^2 - |x_0|^2)(R^2 - |x|^2)}\right| + 2(A - \mathfrak{R}y_0)\frac{R\,|x - x_0|\,|R^2 - \bar{x}_0x|}{(R^2 - |x_0|^2)(R^2 - |x|^2)}\cdot$$

For $x_0 = 0$ the second of these two formulas reduces to a theorem due to Lindelöf (ibid., written however in a simpler form),

$$|y| \leqq \left|y(0) - 2(A - \mathfrak{R}y(0))\frac{|x|^2}{R^2 - |x|^2}\right| + 2(A - \mathfrak{R}y(0))\frac{R\,|x|}{R^2 - |x|^2}\cdot$$

Carathéodory's theorem follows a fortiori from this theorem also.

THEOREM 4. *Under the same assumptions as in theorem* 1, *and* η *being a constant such that* $\eta \neq \eta_0$ *and* $\mathfrak{R}(\eta) < A$,[*] *we have*

$$y \neq \eta \qquad \textit{for} \qquad \kappa = \kappa_0\kappa_1\cdots\kappa_n < \left|\frac{y_0 - \eta}{y_0 + \bar{\eta} - 2A}\right|\cdot[\dagger]$$

PROOF. We consider the function

$$\frac{y - \eta}{y + \bar{\eta} - 2A} = \frac{y - \eta}{- \quad \eta - 2(A - \mathfrak{R}\eta)}$$

which is holomorphic for $|x| < R$ since $\mathfrak{R}(y + \bar{\eta} - 2A) < 0$ by our assumptions. Writing η for y_0 in the proof of theorem 1 we see at once that the function is < 1 in absolute value, and that it takes the value

$$\frac{y_0 - \eta}{y_0 + \bar{\eta} - 2A} \neq 0$$

---

[*] For $\mathfrak{R}(\eta) \geqq A$ we have $y \neq \eta$ in the entire given region $|x| < R$.

[†] This condition may also be written

$$\kappa^2 < \frac{|y_0 - \eta|^2}{|y_0 - \eta|^2 + 4(A - \mathfrak{R}y_0)(A - \mathfrak{R}\eta)}$$

for $x = x_0, x_1, \cdots, x_n$. By reason of § 3 theorem 4 (in which we replace η by 0, y by the function just considered, and M by unity), our theorem is proved. (Another but less simple proof is obtained by applying the α method to the upper inequality in theorem 3. The details of this are left to the reader.)

Corollary. For $\kappa = \kappa_0$ we have

$$y \neq \eta \quad \text{when} \quad \left|\frac{R(x - x_0)}{R^2 - \bar{x}_0 x}\right| < \left|\frac{y_0 - \eta}{y_0 + \bar{\eta} - 2A}\right|,$$

or equivalently (by lemma 4, writing, x for u, x_0 for u_0, $\left|\dfrac{y_0 - \eta}{y_0 + \bar{\eta} - 2A}\right|$ for k, where $|\,(y_0 - \eta)/(y_0 + \bar{\eta} - 2A)\,|\,|\,x_0\,| < |\,x_0\,| < R$),

$$y \neq \eta \quad \text{when} \quad \left| x - x_0 \frac{R^2(|\,y_0 + \bar{\eta} - 2A\,|^2 - |\,y_0 - \eta\,|^2)}{R^2\,|\,y_0 + \bar{\eta} - 2A\,|^2 - |\,x_0(y_0 - \eta)\,|^2} \right|$$
$$< R\,\frac{(R^2 - |\,x_0\,|^2)\,|\,y_0 + \bar{\eta} - 2A\,|\,|\,y_0 - \eta\,|}{R^2\,|\,y_0 + \bar{\eta} - 2A\,|^2 - |\,x_0(y_0 - \eta)\,|^2},$$

and in particular

$$y \neq 0 \quad \text{when} \quad \left| x - x_0 \frac{R^2(|\,y_0 - 2A\,|^2 - |\,y_0\,|^2)}{R^2\,|\,y_0 - 2A\,|^2 - |\,x_0 y_0\,|^2} \right|$$
$$< R\,\frac{(R^2 - |\,x_0\,|^2)\,|\,y_0 - 2A\,|\,|\,y_0\,|}{R^2\,|\,y_0 - 2A\,|^2 - |\,x_0 y_0\,|^2}.$$

When we are satisfied with a smaller region, we have the simpler proposition:

$$y \neq \eta \quad \text{when} \quad |\,x - x_0\,| < \frac{(R^2 - |\,x_0\,|^2)\,|\,y_0 - \eta\,|}{R\,|\,y_0 + \bar{\eta} - 2A\,| + |\,x_0(y_0 - \eta)\,|},$$

as is readily found by replacing $|\,R^2 - \bar{x}_0 x\,|$ a fortiori by $R^2 - |\,x_0\,|^2 - |\,x_0(x - x_0)\,|$ in the condition given at the beginning.

Theorem 5. *Under the same assumptions as in theorem* 1, *we have*

$$-\,(A - \Re y_0)\frac{2\kappa}{1 - \kappa} \leqq \Re(y - y_0) \leqq (A - \Re y_0)\frac{2\kappa}{1 + \kappa}$$

or

$$-\,A\frac{2\kappa}{1 - \kappa} + \Re(y_0)\frac{1 + \kappa}{1 - \kappa} \leqq \Re(y) \leqq A\frac{2\kappa}{1 + \kappa} + \Re(y_0)\frac{1 - \kappa}{1 + \kappa}.$$

Proof. From theorem 1, II it follows a fortiori that

$$(\Re(y - y_0))^2 = (a_0 - a)^2 \leqq 4a_0 a\frac{\kappa^2}{1 - \kappa^2},$$

or equivalently $(a_0 - a)^2 \leqq (a_0 + a)^2\kappa^2$, or

$$\kappa \geqq \frac{|a - a_0|}{a + a_0},$$

or

$$\kappa \geqq \frac{a - a_0}{a + a_0} \geqq -\kappa, \qquad \text{or} \qquad a_0\frac{1-\kappa}{1+\kappa} \leqq a \leqq a_0\frac{1+\kappa}{1-\kappa},$$

which are equivalent to the inequalities to be proved.

NOTE 1. For $\kappa = 0$, both inequalities reduce to identities. Supposing the inequalities to be true for a certain function y, it follows conversely from the inequalities to the right that $\mathfrak{R}(y) = \mathfrak{R}(y_0)$ for $\kappa = 0$, and moreover we have a fortiori $\mathfrak{R}(y - y_0) < A - \mathfrak{R}(y_0)$ or $\mathfrak{R}(y) < A$, provided there exists an x_0 for which $\mathfrak{R}(y_0) < A$.

NOTE 2. For $\kappa = \kappa_0$ we have

$$-(A - \mathfrak{R}y_0)\frac{2R\,|\,x - x_0\,|}{|R^2 - \bar{x}_0x\,| - R\,|\,x - x_0\,|} \leqq \mathfrak{R}(y - y_0)$$
$$\leqq (A - \mathfrak{R}y_0)\frac{2R\,|\,x - x_0\,|}{|R^2 - \bar{x}_0x\,| + R\,|\,x - x_0\,|},$$

which reduces for $x_0 = 0$ to a formula given by Lindelöf (l.c.[11], p. 15):

$$-(A - \mathfrak{R}y(0))\frac{2\,|\,x\,|}{R - |\,x\,|} \leqq \mathfrak{R}(y - y(0)) \leqq (A - \mathfrak{R}y(0))\frac{2\,|\,x\,|}{R + |\,x\,|}.$$

COROLLARY. When $A' < \mathfrak{R}(y) < A$, then

$$(A' - \mathfrak{R}y_0)\frac{2\kappa}{1+\kappa} \leqq \mathfrak{R}(y - y_0) \leqq (A - \mathfrak{R}y_0)\frac{2\kappa}{1+\kappa}.$$

PROOF by applying the inequality to the right, which we know already from theorem 5, to the function $-y$, since $\mathfrak{R}(-y) < -A'$.

THEOREM 6. *Under the same assumptions as in theorem* 1, *we have*

$$|\,\mathfrak{I}(y - y_0)\,| \leqq 2(A - \mathfrak{R}y_0)\frac{\kappa}{1-\kappa^2}.$$

PROOF follows at once from theorem 1, III by replacing a fortiori

$$y - y_0 + 2a_0\frac{\kappa^2}{1-\kappa^2}$$

on the left by its imaginary part.*

NOTE 1. For $\kappa = 0$ it follows from the inequality that $\mathfrak{I}(y) = \mathfrak{I}(y_0)$. The inequalities in theorems 5 and 6 taken together give again all the assumptions from which we started, viz., $y(x_\nu) = y(x_0)$ for $\nu = 1, 2, \cdots, n$ and $\mathfrak{R}(y) < A$, provided that there exists an x_0 for which $\mathfrak{R}(y_0) < A$.

---

* In the particular case when x is known to have such a value that $\mathfrak{R}(y) \geqq \mathfrak{R}(y_0)$, theorem 1, II gives the stronger inequality

$$|\,\mathfrak{I}(y - y_0)\,| \leqq 2(A - \mathfrak{R}y_0)\frac{\kappa}{\sqrt{1-\kappa^2}}.$$

NOTE 2. For $\kappa = \kappa_0$, we have

$$|\Im(y - y_0)| \leqq 2(A - \Re y_0)\frac{R\,|x - x_0|\cdot|R^2 - \bar{x}_0 x|}{(R^2 - |x_0|^2)(R^2 - |x|^2)},$$

which gives, for $x_0 = 0$, another formula due to Lindelöf (l.c.):

$$|\Im(y - y(0))| \leqq 2(A - \Re y(0))\frac{R\,|x|}{R^2 - |x|^2}.$$

THEOREM 7, CONCERNING THE DERIVATIVE OF y. *Under the same assumptions as in theorem 1, we have*

$$\left|\frac{dy}{dx}\right| \leqq 2R\frac{A - \Re(y)}{R^2 - |x|^2} \leqq 2R\frac{A - \Re(y_0)}{R^2 - |x|^2}\,\frac{1 + \kappa}{1 - \kappa}.^*$$

PROOF. The first inequality is already known from note 2 to theorem 2. The second inequality follows by replacing $a = A - \Re(y)$ a fortiori by

$$a_0\frac{1 + \kappa}{1 - \kappa} = (A - \Re y_0)\frac{1 + \kappa}{1 - \kappa}$$

in view of the last inequality in the proof of theorem 5.

NOTE 1. For $\kappa = 0$, the second and third members are identical; the second member may therefore be removed without making the formula less general.

NOTE 2.† To find when the equality sign holds in this theorem, let us assume that for $x = x_0$ and $y = y_0$ ($|x_0| < R$, $\Re(y_0) < A$) we have

$$\left|\frac{dy}{dx}\right|_0 = 2R\frac{A - \Re(y_0)}{R^2 - |x_0|^2}.$$

The function

$$u = \frac{y - y_0}{y - y_0 - 2a_0}\,\frac{R^2 - \bar{x}_0 x}{R(x - x_0)}$$

is holomorphic for $|x| < R$, and from theorem 1, I with $\kappa = \kappa_0$, we find $|u| \leqq 1$ for $|x| < R$. Moreover,

$$u(x_0) = \lim_{x \to x_0} u = -\frac{R^2 - |x_0|^2}{2R(A - \Re y_0)}\lim_{x \to x_0}\frac{y - y_0}{x - x_0} = -\frac{R^2 - |x_0|^2}{2R(A - \Re y_0)}\left(\frac{dy}{dx}\right)_0,$$

whence $|u(x_0)| = 1$. By the argument used in § 3, theorem 1, note 2, it

---

* When it is known moreover that $\Re(y) > A'$, we have a fortiori

$$\left|\frac{dy}{dx}\right| < 2R\frac{A - A'}{R^2 - |x|^2}.$$

† This note has been changed in the translation to correspond to the changes in the note to lemma 6.

follows that u is a constant $= -\gamma$, where $|\gamma| = 1$, and consequently

$$y = y_0 + 2\gamma(A - \Re y_0)\frac{R(x - x_0)}{R^2 - \bar{x}_0 x + \gamma R(x - x_0)}.$$

Conversely, it is seen that for any y of this form, where $|x_0| < R$, $\Re(y_0) < A$, $|\gamma| = 1$, the equality sign holds to the left in our theorem when $x = x_0$.

NOTE 3. Writing y as a power series in x

$$y(x) = c_0 + c_1 x + c_2 x^2 + \cdots,$$

(with a radius of convergence $\geqq R$), the first inequality in theorem 7 gives the relation

$$R\,|c_1| \leqq 2(A - \Re c_0)$$

for $x_0 = 0$, as noted by Lindelöf (l.c.). As in note 2 to theorem 5 in § 3 we may derive from this the more general relation

$$R^q\,|c_q| \leqq 2(A - \Re c_0),$$

q being a positive integer.

THEOREM 8, CONCERNING THE DIFFERENCE QUOTIENT OF y. *Under the same assumptions as in theorem* 1, *we have for* $x \neq x^*$

$$\left|\frac{y - y^*}{x - x^*}\right|^2 \leqq 4R^2\frac{(A - \Re y)(A - \Re y^*)}{(R^2 - |x|^2)(R^2 - x^{*2})},$$

x^* *being a new independent variable, and* $y^* \equiv y(x^*)$. *When conversely this inequality is satisfied, and there exists a certain value of* x *for which* $\Re(y) < A$, *then* $\Re(y) < A$ *for all* x.

PROOF. Theorem 1, II, states for $\kappa = \kappa_0$ that the inequality

$$|y - y_0|^2 \leqq 4a_0 a\frac{\kappa_0^2}{1 - \kappa_0^2} = |x - x_0|^2 \cdot 4R^2\frac{a_0 a}{(R^2 - |x|^2)(R^2 - |x_0|^2)}$$

is complete when an x_0 exists such that $\Re(y_0) < A$. Replacing x_0 by x^*, our theorem is proved.

NOTE. When it is desired to avoid y or y^* to the right in the inequality in theorem 8, we replace a fortiori (as in the preceding theorem)

$$A - \Re(y) \quad \text{by} \quad (A - \Re y_0)\frac{1 + \kappa}{1 - \kappa}$$

and

$$A - \Re(y^*) \quad \text{by} \quad (A - \Re y_0)\frac{1 + \kappa^*}{1 - \kappa^*}.$$

Exactly as in note 2 to theorem 6 in § 3, this may be varied in several ways.